Real-fibered morphisms of real del Pezzo surfaces

Joint with Mario Kummer and Cédric Le Texier

X non-sing. alg variety

$\dim X = n$

$\sigma : X \longrightarrow X$ real structure

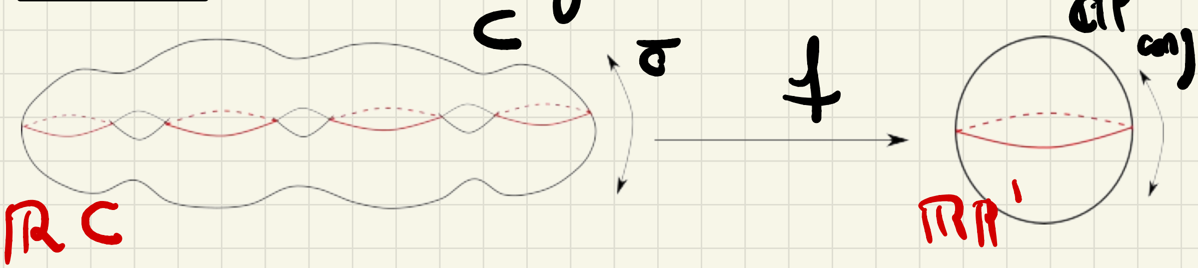
$\mathrm{fix}(\sigma) = \mathbb{R}X$ real part

DEF: $f : \overset{\circlearrowleft\,\sigma}{X} \longrightarrow \overset{\circlearrowleft\,\mathrm{conj}}{\mathbb{CP}^n}$ real morphism. We say f is <u>real-fibered</u> if

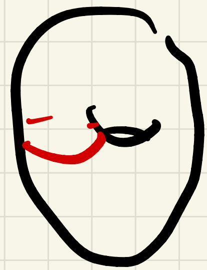
$$f^{-1}(\mathbb{RP}^n) = \mathbb{R}X$$

- (X, σ) $\quad \mathbb{R}X \neq \emptyset$

m=1 Real algebraic curves



ex. $g=1$ real curve



TH (Ahlfors '50): $\exists\, f : C \to \mathbb{CP}^1$ real-fibered iff $\underbrace{\mathbb{C}C \setminus \mathbb{R}C \text{ is disconnected}}_{C \text{ is separating}}$.

$\iff [\mathbb{R}C]$ trivial $H_1(C;\mathbb{Z}/2\mathbb{Z})$

• $\forall\, (C,\sigma)$ separating

1) $\boxed{\ell \equiv g(C)+1 \mod 2}$ $\quad \ell = \#$ c.c. of $\mathbb{R}C$

2) $\ell = g(C)+1$ maximum for ℓ

$\rightsquigarrow$ C is <u>maximal</u>

· Harnack-Klein inequality

Any maximal cre is separating

- Rokhlin, Klein, Arnold CITE Kummer-Shaw, Orevkov-Mikhalkin, Gabard ...

$\boxed{\text{From } n=1 \text{ to } n \geq 2}$

$\boxed{n=1}$

$\exists\, C \to \mathbb{C}P^1$ real fibered | C is separating | $[\mathbb{R}C]$ trivial $H_1(C;\mathbb{Z}/2)$

$\boxed{n \geq 2}$

(X,σ) $\exists f : X \to \mathbb{C}P^n$

Can we characterize X?

$\cancel{(\Leftrightarrow)}$

VIRO'S

(X,σ) bounding in complexific.

$[\mathbb{R}X]$ trivial in $H_n(X;\mathbb{Z}/2\mathbb{Z})$

VIRO $\not\Rightarrow$ $\exists$ REAL FIBERED

$\exists$ REAL FIBERED $\not\Rightarrow$ VIRO

$\boxed{m \geq 2}$ RIGIDITY

(X, σ) $\quad \exists\, f : X \longrightarrow \mathbb{CP}^m$ real fibered

1) Kummer - Shamovic '18

$f|_{\mathbb{R}X} \; \mathbb{R}X \longrightarrow \mathbb{RP}^n$ <u>unramified</u>

$$\boxed{\mathbb{R}X \simeq \bigsqcup_{s} S^m \sqcup \bigsqcup_{r} \mathbb{RP}^m}$$

2) $\deg f = 2s + r$

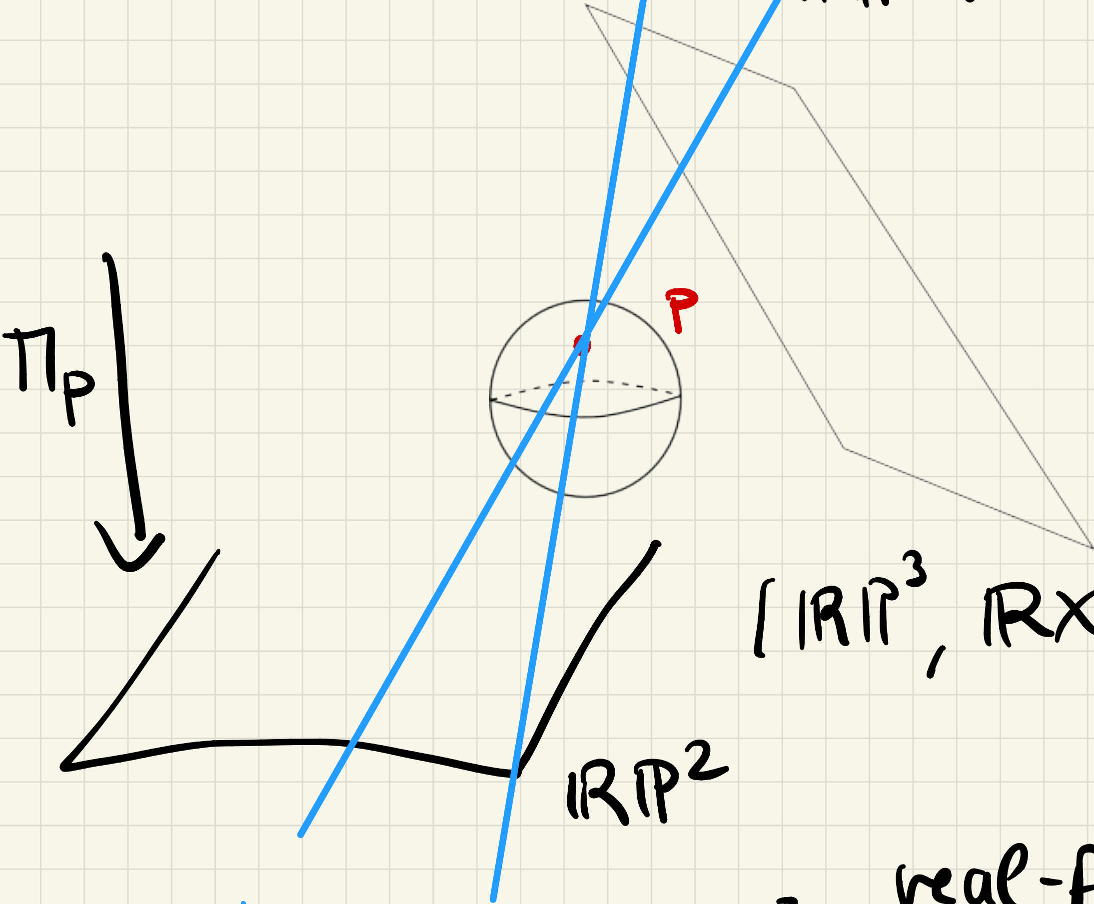
<u>Rk</u>: (X, σ) $\exists f : X \longrightarrow \mathbb{CP}^m$ real fibered

$\tilde{f} : Bl_{p_1, p_2}(X) \longrightarrow \mathbb{CP}^m$ real fibered non-finite

p_1, p_2 cx conj

<u>example</u>: X cubic surface $\subseteq \mathbb{CP}^3$

$\mathbb{R}X = S^2 \sqcup \mathbb{RP}^2$



$\Pi_p|_X : X \dashrightarrow \mathbb{CP}^2$ real-fib. of deg 3

DEF: $f: X \to \mathbb{CP}^m$ real-fibered if $f: X \xhookrightarrow{i} \mathbb{CP}^n \overset{\pi_E}{\dashrightarrow} \mathbb{CP}^m$

we say f is <u>hyperbolic</u> wrt E.

(X is <u>hyperbolic</u>)

- real very ample divisor on $X \xhookrightarrow{i} \mathbb{CP}^n$
- E lin sub $\subset \mathbb{CP}^n$ $\operatorname{codim} E = m+1$
  $E \cap X = \emptyset$ $\pi_E|_X$ $X \to \mathbb{CP}^m$ real-fib

<u>Remark</u>: $\boxed{m=1}$ <u>Kummer - Shaw</u>

- Any separating curve admits a hyperbolic morphism.

- There are real-fibered morphs. which are not hyperbolic.

Del Pezzo surfaces

- real classif. (Comessatti 1914)

- there are (X, ε) del Pezzo
$$\mathbb{R}X \cong \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$$

DEF: X non-sing alg surface
X is del Pezzo if $-K_X$ ample

$$\deg X = -K_X^2 = d \qquad 0 \leq d \leq 9$$

RK : • $d=9$ $X=\mathbb{CP}^2$ $\mathbb{R}X=\mathbb{RP}^2$

• For some degrees d, $\forall X$ of deg d there are some $\sigma: X \to X$ s.t. $\mathbb{R}X \neq \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$

• $1 \leq d \leq 7$ there are (X, σ) s.t. $\mathbb{R}X$ has the right topology admits only real-fibered morphism $f: X \to \mathbb{CP}^2$ which are non-finite

TH (KLTM) 1: the classification of finite real-fibered morphisms of real del Pezzo surface X of deg d with $\mathbb{R}X \simeq \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$ is as follows:

(X,σ) esmt: one $\mathbb{R}X \cong \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$ $d = \deg X$

X	d	s	r	# real-fibered	HYPERBOLIC	h
quadric ellipsoid	8	1	0	1 $-K_X$	✓	
X $p \in \mathbb{R}X$	4	2	0	1 $-K_X$	✓	4
$Bl_p X$ $q \in \mathbb{R}Bl_p X$	3	1	1	1 $-K_X$	✓	3
$Bl_{p,q} X$	2	0	2	1 $-K_X$	✗	
	2	3	0	2 GEYSER	✓	5
	2	4	0	1 $-2K_X$	✓	6
	1	2	1	4 2 pairs BERTINI	✓	4
	1	3	1	2 BERTINI	✓	5
	1	4	1	1 $-3K_X$	✓	6

<u>Hyperbolic</u>: $X \xhookrightarrow{i} \mathbb{CP}^h \overset{\pi_E}{\dashrightarrow} \mathbb{CP}^2$

$h = s + 2$ $r \in \{0, 1\}$

- $\deg X = 2$ $\quad \mathbb{R}X = \mathbb{R}\mathbb{P}^2 \sqcup \mathbb{R}\mathbb{P}^2$

$f : X \xrightarrow{2-1} \mathbb{C}\mathbb{P}^2 \hookleftarrow Q$ non-sing. quartic

$\mathbb{R}Q = \emptyset$

Main ingredients of the proof

(X, σ) del Pezzo surf. $\mathbb{R}X = \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$

admits a finite $f: X \to \mathbb{CP}^2$

real fibered $\rightsquigarrow$ ample real D asś. to f

- $r \leq D \cdot K_X + 4 \leq 2s + r$

if $r=0$ $1 \leq$

- $r \equiv D \cdot K_X \bmod 4$

RK: $f^{-1}(L) = C$ a general line in $\mathbb{CP}^2$ is separating curve in X $s+r$ c.c. in $\mathbb{R}C$

$$s + r \equiv g(C) + 1 \bmod 2$$

(DEL PEZZO)

COROLLARY: $\exists$ # finite of candidates real ample divisor D on X which may give a $f: X \to \mathbb{CP}^2$ real fibered

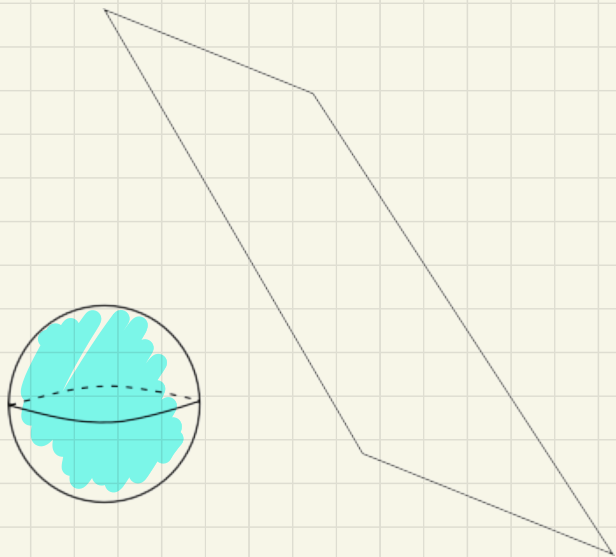
Reider method

- brute force
- D very ample and real (di Rocco 96)
- Find $E \subset \mathbb{CP}^4$ s.t. $E \cap X = \emptyset$ and $\pi_E|_X : X \longrightarrow \mathbb{CP}^2$ real-fib.

$+ \ \dots$

Back to cubic surface example

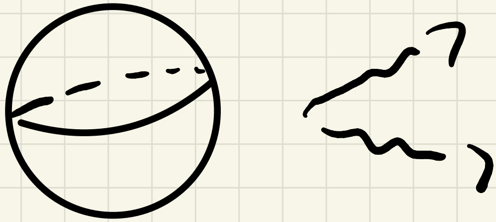
$X \subset \mathbb{CP}^3$



$(\mathbb{RP}^3, \mathbb{R}X)$

Need of such a criterion

ex of cubic surface:



in $\mathbb{RP}^3$

$X \subseteq \mathbb{CP}^n$

A way to assure that can constr.
a codim 3 lin. sub. E s.t.
$H^{n-2} \supseteq E$ $\quad H^{n-2} \cap X$ in
only real pts.

DEF: $K, L \in S^n$ K m-sphere, L ℓ-sphere

$n = \underline{m + \ell + 1}$

$[K], [L]$ W chain

$\partial W = K$

LINKING NUMBER $\ell k(K, L) = \# W \cap L$

LINKING NUMBER

DEF: $X \subseteq \mathbb{CP}^n$ $\mathbb{R}X = X_1 \sqcup \ldots \sqcup X_{s+r}$

$\dim X = m$ $\mathbb{R}X = \bigsqcup_s S^m \sqcup \bigsqcup_r \mathbb{RP}^m$

E lin sub. $\subseteq \mathbb{CP}^n$, $E \cap X \neq \emptyset$ codim $m+1$

$\pi : S^n \xrightarrow{2-1} \mathbb{RP}^n$

$\ell k(X_i, \mathbb{R}E) = \ell k(\underbrace{\pi^{-1}(X_i)}_{K_1 \sqcup K_2}, \pi^{-1}(\mathbb{R}E)$

$\ell + m + 1 = n$

TH (KLTM) 2:

$X \subseteq \mathbb{CP}^n$, $E \subseteq \mathbb{CP}^n$ as above $\deg X = 2s + r$. Then X is hyperbolic wrt E iff

$$\boxed{2s+r = \sum_{i=1}^{s+r} |ek(X_i, \mathbb{R}E)|}$$

$\boxed{m=1}$ Rummer-Shaw

- $ek(S^2, \mathbb{R}E) = \pm 2$
- $r = 0, 1$ $ek(\mathbb{RP}^2, \mathbb{R}E) = \pm 1$

TH(KLTM)3: $X \subset \mathbb{CP}^n$ surface
$\mathbb{R}X \simeq \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$ $n = s+2$
$r \in \{0,1\}$. Assume $g(H \cap X)$
is $s+r-1$ for a generic
hyperpl. H in $\mathbb{CP}^n$. Then $\exists$
E h.m. sub. $\subset \mathbb{CP}^n$ s.t. X is hyperbolic
wrt E.

<u>Proof</u>: Since $n > s$
$\exists$ H hyp. $\mathbb{CP}^n$ s.t. $H \cap X = C$
non-sing real curve s.t.

- # of c.c. of $\mathbb{R}C = s+r$
- C is separating
- Each c.c. of $\mathbb{R}X$ contains exacty one c.c. of $\mathbb{R}C$

Kummer-Shaw $\longrightarrow$ Can construct E lin sub. $\subseteq \mathbb{H}$ s.t.

C is hyperbolic wrt E in $\mathbb{H}$

TA2 $\Rightarrow$ X is hyperbolic wrt E in $\mathbb{CP}^n$.

LINKING #

- End Proof THM 1

Question:

$X \quad \deg X = 2$ $\qquad \sigma_1 \circlearrowright X \quad \mathbb{R}X_1 = \bigsqcup_{4} S^2$

$\qquad\qquad\qquad \sigma_2 \circlearrowright X \quad \mathbb{R}X_2 = \bigsqcup_{3} S^2$

Both $\longrightarrow$ hyperbolic morphism

$[\mathbb{R}X_1]$ trivial in $H_1(X; \mathbb{Z}/2\mathbb{Z})$

$[\mathbb{R}X_2]$ not trivial in $H_1(X; \mathbb{Z}/2\mathbb{Z})$

(X, σ)

$[\mathbb{R}X] = 0$

$$Y \xrightarrow{2-1} X \longleftrightarrow \mathcal{L} = \mathcal{L}(D)$$

Kummer-Shamovic:

$f : X^m \longrightarrow Y^m$ real-fibered, $\sigma \circlearrowright X$, $\tau \circlearrowright Y$

$f|_{\mathbb{R}X} : \mathbb{R}X \longrightarrow \mathbb{R}Y$ unramified.

EXAMPLE of a diff. construction of E

(X, σ) del Pezzo $\deg X = 2$

$\mathbb{R}X = \bigsqcup_4 S^2$

$$\exists f : X \xhookrightarrow[|-2K_X|]{i} \mathbb{CP}^6 \overset{\pi_E}{\dashrightarrow} \mathbb{CP}^2$$

CONSTRUCT E by hand:

We want $\pi_E|_X : X \to \mathbb{CP}^2$ real fibered

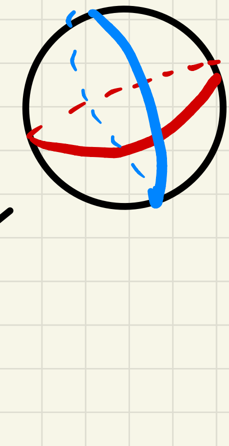
$(\equiv)$

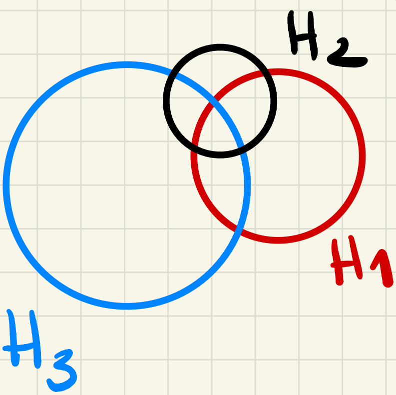
$E =$ hyp. in $\mathbb{CP}^6$

E of codim 3

F

$(\quad , \quad)$





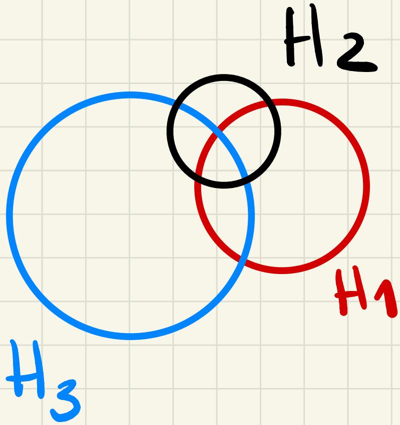
Then X

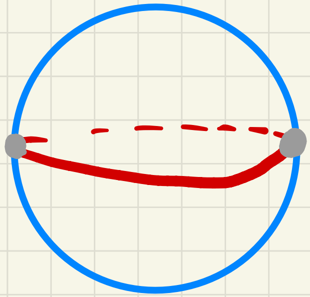
Pencils of HYPERPLANES :

$\Pi_{H_1 H_3}$ $\Pi_{H_1 H_2}$

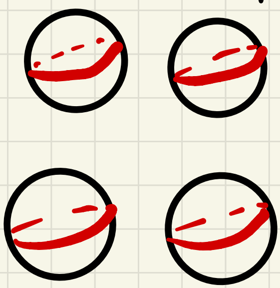
<u>Rk</u> : Any $\mathbb{CP}^4$ in $\mathbb{CP}^6$ containing $E = H_1 \cap H_2 \cap H_3$

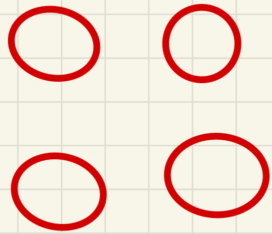
- Any pair (H_{13}, H_{12})


H2
H1
H3



HINT : $\sigma \circlearrowright X \xrightarrow[|1-K_X|]{} \overset{\& \text{ conj}}{\mathbb{CP}^2} \hookleftarrow C$ deg 4 curve

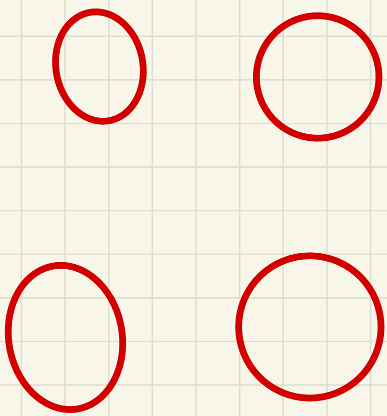




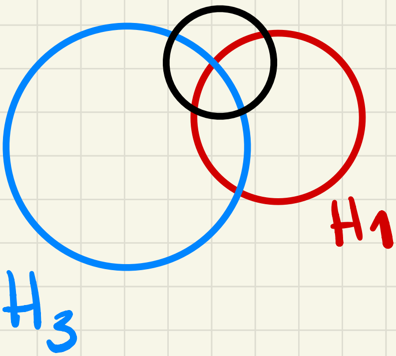
$(\mathbb{RP}^2, \mathbb{R}C)$

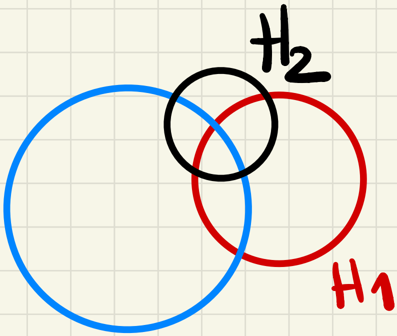
•

Two conics A_1, A_2 s.t.



$\left(\mathbb{RP}^2, \bigsqcup_{i=1}^{2} \mathbb{R}A_i \sqcup \mathbb{R}C\right)$


H1
H3


H2
H1